BackPaper

Class Field Theory

Instructor: Ramdin Mawia **Marks:** 45 **Time:** June 07, 2023; 10:00–13:00.

INSTRUCTIONS

i. Attempt THREE problems, including problem n° 5. The marks are indicated against each question. The maximum you can score is **45**.

ii. You may use any of the results proved in class, unless you are asked to prove or justify the result itself. You may also use results from other problems in this question paper, provided you attempt and correctly solve the problem.

iii. The notation is standard: $\mathcal{O}_F$ denotes the valuation ring of a nonarchimedean local field F, and k_F denotes its residue field. Also, $\mathcal{C}_K$ denotes the class group of a global field K, and $\mathbb{A}_K^\times, \mathbb{A}_K^1$ denote the group of idèles and 1-idèles of K.

Class Field Theory

1. State the Dirichlet Unit Theorem for number fields and prove it using the theory of adèles and idèles. **20**

2. Let $\mathfrak{p}$ be a finite prime of a number field K, and let L/K be a finite Galois extension. Prove that, for any prime $\mathfrak{q}$ of L lying above $\mathfrak{p}$, the extension $L_\mathfrak{q}/K_\mathfrak{p}$ is Galois with Galois group $G_\mathfrak{q}$, the decomposition group of $\mathfrak{q}$. **20**

3. Let K be a number field. Prove the following group isomorphisms: **20**

$$\mathcal{C}_K \cong K^\times \mathbb{A}_{K,\infty}^\times \backslash \mathbb{A}_K^\times \cong K^\times \mathbb{A}_K^1 \backslash \mathbb{A}_K^1.$$

Also, prove that the last group above is finite, and hence so are the other two groups. The notations are standard. Be as complete as you can.

4. Let E/F be a finite extension of p-adic fields, and let k_E, k_F be the residue fields of E, F respectively. Prove that the extension E/F is unramified if and only if there exists an $x \in E$ satisfying the following: **20**

   i. The minimal polynomial $f(X)$ of x over F has all its coefficients in $\mathcal{O}_F$, the valuation ring of F.

   ii. The reduction $\bar{f}(X) \in k_F[X]$ is irreducible.

   iii. $E = F[x]$.

   In this case, prove that $\mathcal{O}_E = \mathcal{O}_F[x]$.

5. State true or false, with brief but complete justifications (**any two**): **10**

   i. Any finite Galois extension of p-adic fields is solvable.

   ii. Let $K = \mathbb{Q}[\sqrt{2}]$. Then the unit group $\mathcal{O}_K^\times$ of the number field K is a finitely generated abelian group of rank 2.

   iii. For any finite extension L/K of number fields, the norm morphism $\mathrm{N}_{L/K} : L^\times \to K^\times$ is surjective.